MATH 1650: GRAPHS OF POLYNOMIAL FUNCTIONS

EXPLORATION: Use a graphing utility to graph the following functions. What patterns evolve?

If n is even, and $a > 0$, the graph of $f(x) = ax^n$ looks like: an upward facing '$\cup$' shape.

If n is even, and $a < 0$, the graph of $f(x) = ax^n$ looks like: a downward facing '$\cap$' shape.

If n is odd, and $a > 0$, the graph of $f(x) = ax^n$ looks like: an increasing 'S' shape.

If n is odd, and $a < 0$, the graph of $f(x) = ax^n$ looks like: a decreasing 'S' shape.

EXAMPLE: For $f(x) = 1 - 2x - 4x^3 = (-4)x^3 + 0x^2 + (-2)x + 1$:

$a_3 = -4$ $\qquad$ $a_2 = 0$ $\qquad$ $a_1 = -2$ $\qquad$ $a_0 = 1$

TERMINOLOGY: A polynomial function: $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$, with $a_n \neq 0$ has:

- **degree** n
- **leading term** $a_n x^n$
- **leading coefficient** a_n

EXAMPLE: Find the degree, leading term, and leading coefficient of the following polynomial functions.

- $f(x) = -2x^4 + 5x^3 - 9x^2 + 3$
  - degree: 4
  - leading term: $-2x^4$
  - leading coefficient: -2
- $g(x) = 3 - 2x - x^3 = -x^3 - 2x + 3$
  - degree: 3
  - leading term: $-x^3$
  - leading coefficient: -1
- $F(x) = (2x - 1)(x^2 + 3x + 1) = 2x^3 + \ldots$
  - degree: 3
  - leading term: $2x^3$
  - leading coefficient: 2
- $G(x) = -2(x-1)^3(x+2)^2 = -2x^5 + \ldots$
  - degree: 5
  - leading term: $-2x^5$
  - leading coefficient: -2
- $p(z) = -(3z-1)(z+2)^2\left(z^2+1\right) = -3z^5 + \ldots$
  - degree: 5
  - leading term: $-3z^5$
  - leading coefficient: -3
- $q(z) = \frac{1}{2}(2z+3)^2(z-1)^3 = 2z^5 + \ldots$
  - degree: 5
  - leading term: $2z^5$
  - leading coefficient: 2

END BEHAVIOR:

EXAMPLE: Use a graphing utility to graph each of the functions below to determine their end behavior.

- $f(x) = x^3 - 4x + 1$
  - as $x \to -\infty$, $f(x) \to -\infty$
  - as $x \to \infty$, $f(x) \to \infty$
- $g(t) = -t^4 + 2t^2 - 1$
  - as $t \to -\infty$, $g(t) \to -\infty$
  - as $t \to \infty$, $g(t) \to -\infty$

Graph each function above along with its leading term on a graphing utility and 'Zoom Out.' What happens?
The graph of the polynomial function looks more or less like the graph of the leading term.

LEADING TERM TEST: The end behavior of a polynomial matches the end behavior of its leading term.

EXAMPLE: Use the leading term test to predict the end behavior of the following functions.
Check your answer using a graphing utility.

- $F(x) = 100x^2 - x^3 = -x^3 + 100x^2$
  - leading term: $-x^3$
  - as $x \to -\infty$, $F(x) \to \infty$
  - as $x \to \infty$, $F(x) \to -\infty$
- $G(x) = 0.01x^4 - 100x^3 + 1$
  - leading term: $0.01x^4$
  - as $x \to -\infty$, $G(x) \to \infty$
  - as $x \to \infty$, $G(x) \to \infty$
- $h(t) = (t+1)(2-t)(t+3) = -t^3 + \ldots$
  - leading term: $-t^3$
  - as $t \to -\infty$, $h(t) \to \infty$
  - as $t \to \infty$, $h(t) \to -\infty$
- $p(x) = 3(x-5)^2(x^2+x+1) = 3x^4 + \ldots$
  - leading term: $3x^4$
  - as $x \to -\infty$, $p(x) \to \infty$
  - as $x \to \infty$, $p(x) \to \infty$
- $p(z) = (1-z)^3(2z+1) = -2z^4 + \ldots$
  - leading term: $-2z^4$
  - as $z \to -\infty$, $p(z) \to -\infty$
  - as $z \to \infty$, $p(z) \to -\infty$
- $q(x) = 2(x-1)(3-2x)^2 = 8x^3$
  - leading term: $8x^3$
  - as $x \to -\infty$, $q(x) \to -\infty$
  - as $x \to \infty$, $q(x) \to \infty$

ZEROS OF POLYNOMIALS:

DEFINITION: The **zeros** of a function f are the solutions to the equation $f(x) = 0$.

NOTE: Geometrically, c is a zero of f means $(c, 0)$ is an x-intercept of the graph of $y = f(x)$.

EXAMPLE: All of the functions below have the same zeros: $x = 0$, $x = 1$, and $x = -2$. For each function, state the multiplicity of each zero and describe what the graph looks like 'near' the corresponding x-intercept. Choose among the adjectives: linear, 'u'-shaped, 's'-shaped. What patterns do you notice?

- $f(x) = x(x-1)(x+2)$

zero	multiplicity	shape
$x = 0$	1	linear
$x = 1$	1	linear
$x = -2$	1	linear

- $f(x) = x^2(x-1)(x+2)$

zero	multiplicity	shape
$x = 0$	2	'u'-shaped
$x = 1$	1	linear
$x = -2$	1	linear

- $f(x) = x(x-1)^2(x+2)$

zero	multiplicity	shape
$x = 0$	1	linear
$x = 1$	2	'u'-shaped
$x = -2$	1	linear

- $f(x) = x^2(x-1)(x+2)^2$

zero	multiplicity	shape
$x = 0$	2	'u'-shaped
$x = 1$	1	linear
$x = -2$	2	'u'-shaped

- $f(x) = x^3(x-1)(x+2)^2$

zero	multiplicity	shape
$x = 0$	3	's'-shaped
$x = 1$	1	linear
$x = -2$	2	'u'-shaped

- $f(x) = x^4(x-1)(x+2)^3$

zero	multiplicity	shape
$x = 0$	4	'u'-shaped
$x = 1$	1	linear
$x = -2$	3	's'-shaped

If $x = c$ is a zero of f with multiplicity **one**, the graph of $y = f(x)$ is <u>linear</u> near $(c, 0)$.

If $x = c$ is a zero of f with **even** multiplicity, the graph of $y = f(x)$ is <u>'u'-shaped</u> near $(c, 0)$.

If $x = c$ is a zero of f with **odd** multiplicity larger than 1, the graph of $y = f(x)$ is <u>'s'-shaped</u> near $(c, 0)$.

EXAMPLE: Putting it all together: let $p(x) = -(x+1)^2(x-2)(x^2+1)$.

- Find the leading term of $p(x)$ and use this to describe the end behavior of the graph of $y = p(x)$.

  $p(x) = -(x+1)^2(x-2)(x^2+1) = -(x)^2(x)(x^2) + ... = -x^5 + ...$ So the leading term is $-x^5$.

  As $x \to -\infty$, $p(x) \to \infty$ and as $x \to \infty$, $p(x) \to -\infty$.

- Find all real zeros of p and state their multiplicities.

  To find the zeros of p, we solve $p(x) = 0$.

  Setting $-(x+1)^2(x-2)(x^2+1) = 0$ gives $(x+1)^2 = 0$ or $(x-2) = 0$ or $(x^2+1) = 0$.

  From $(x+1)^2 = 0$, we get $x = -1$ twice.

  From $(x-2) = 0$, we get $x = 2$ once.

  Finally, $(x^2+1) = 0$ gives $x^2 = -1$ so we have no real zeros.

  Hence, our real zeros are $x = -1$ multiplicity 2 and $x = 2$ multiplicity 1.

- Find the x-intercepts of the graph of $y = p(x)$ and describe the behavior of the graph near each x-intercept.

  Since $x = -1$ is a zero of p, the point $(-1, 0)$ is an x-intercept of the graph.

  Since the multiplicity of $x = -1$ is **even**, the graph makes 'u'-shape near $(-1, 0)$.

  Since $x = 2$ is a zero of p, the point $(2, 0)$ is an x-intercept of the graph.

  Since the multiplicity of $x = 2$ is **one**, the graph looks linear near $(2, 0)$.

- Find $p(0)$ and use this to find the y-intercept of the graph of $y = p(x)$.

  Since $p(0) = -((0)+1)^2((0)-2)((0)^2+1) = 2$, the y-intercept is $(0, 2)$.

- Graph $y = p(x)$ by hand using end behavior, the y-intercept, and behavior near the x-intercepts as a guide.

  From the end behavior, we know the graph of $y = p(x)$ starts in Quadrant II.

  The graph bounces at the x-intercept $(-1, 0)$ then heads up to the y-intercept $(0, 2)$.

  The graph the returns to $(2, 0)$ and crosses through into Quadrant IV, which matches the end behavior.

- Use your hand-drawn graph to make a sign diagram for $p(x)$.

  With zeros $x = -1$ and $x = 2$, we know $p(x)$ is $(+)$ when the graph is **above** the x-axis.

  Likewise, $p(x)$ is $(-)$ when the graph of p is **below** the x-axis.

$$
\begin{array}{ccccccccc}
 & (+) & 0 & (+) & 0 & (-) & p(x) \\
-\infty & \longleftarrow & + & \longrightarrow & + & \longrightarrow & \infty \\
 & & -1 & & 2 & & x
\end{array}
$$

- Using desmos confirms our analysis.